An Alternative Proof of Euler's Rotation Theorem

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The study of rigid-body dynamics is an important topic in classical mechanics. As in the case of point particle dynamics, a good understanding of the kinematic description of rigid body motion is essential for understanding its dynamics. One of the crucial results related to the kinematic description of motion of a rigid body is Chasles's theorem [5, 9, 10], which states that a general displacement of a rigid body can be described by a translation and a rotation about an axis. Further, even though the translation vector is not unique, the orientation of the axis of rotation and the angle of rotation will be the same in passing from one configuration to another [2, 14, 16].

This natural splitting of a general motion into translational and rotational parts makes it possible to study the dynamics of translation and rotation separately. The former reduces to a point-particle-like dynamics, and the latter is described using Euler's equations in rigid-body dynamics.

It turns out that one can use the freedom in choosing the translation vector to make its direction coincide with that of the axis of rotation. This is the content of the Mozzi–Chasles theorem and is central to the study of dynamics of rigid bodies using screw theory [1], which is used extensively in present-day robotics [6, 11]. The Mozzi–Chasles theorem was first proved by the Italian mathematician and astronomer Giulio Mozzi in the year 1763 [4, 10]. It was later analyzed by the French mathematician Michel Chasles in his 1830 work [5].

Chasles's theorem can be thought of as an extension of a theorem due to Euler, known as Euler's rotation theorem, which states that every reconfiguration of a rigid body with one of its points fixed is equivalent to a single rotation about an axis passing through the fixed point. In other words, however a sphere might be rotated around its center, a diameter can always be chosen whose direction in the rotated configuration will coincide with that in the original configuration.

The original proof given by Euler is a geometric one [7, 13]. The proof looks at the initial and final configurations of a great circle on the sphere and gives a recipe to construct a point that is subsequently shown to be the point through which the axis of rotation passes. It is interesting to note that Euler's proof of the rotation theorem that bears his name was published in 1776, three years after Mozzi's work [7].

There are various other proofs available for the rotation theorem, both geometric [2, 14] and algebraic [12, 13]. The algebraic proofs typically require a familiarity with rotation matrices and their properties or with ideas from group theory. The most commonly seen analytic proof [3, 8, 15] uses the orthogonality property of three-dimensional rotation matrices to show that they always have an eigenvector with eigenvalue equal to one. This proof makes use of the result that eigenvalues of an orthogonal matrix have modulus one.

Another of the geometric proofs [2] involves looking at the displacement of a segment under rotations and constructs planes of symmetry using the endpoints of the original and displaced segments. The intersection of these symmetry planes is then shown to be the axis of rotation. This is by far the most transparent of the existing proofs.

The proof of Euler's rotation theorem given by Pars [14] involves going from the initial configuration to the final configuration using two rotations: the first a rotation through the angle π about the center of the great circle connecting one of the original points and its final location,

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and the second a rotation about an axis passing through this final location. The invariant point is then established by a construction.

The proof presented here came about as a result of the author's effort to teach Euler's rotation theorem and the Mozzi–Chasles theorem to an undergraduate class. The proof is geometric in nature and is different from the existing ones. It makes use of two successive rotations about two mutually perpendicular axes to go from one configuration of the rigid body to the other with one of its points fixed. We shall first develop the key ideas involved in the description of a rigid body and use this knowledge in setting up the proof of the theorem. After proving Euler's rotation theorem, which is the crux of the paper, Chasles's theorem is derived. The plan of the paper is as follows: We first derive the number of degrees of freedom of a rigid body. We then set up and employ a scheme for describing the general displacement of the rigid body. We then derive Euler's rotation theorem, which we follow with a proof of Chasles's theorem.

To broaden the scope of the discussion and make it more complete, we shall then look at a few consequences of the theorems we have proved involving the idea of screw axis and rigid-body motion in two dimensions.

Rigid-Body Displacement

A rigid body is defined as a collection of particles whose mutual distances remain invariant. In three dimensions, N independent particles have $3N$ degrees of freedom. But if the particles constitute a rigid body, the number of degrees of freedom is reduced to 6 (for the case $N \geq 3$). This is so because the constraint equations that come from the invariant interparticle separations makes $3N - 6$ of the original $3N$ variables dependent on the remaining 6. Let us prove this result rigorously.

We first show that a rigid body configuration is completely defined once coordinates of any three noncollinear particles in the rigid body are specified. Assume that the positions of three particles (A, B, C) are given. Consider now a fourth particle, D. Since the body is rigid, the distances between particles A and D (say d_1), B and D (d_2), and C and D (d_3) are fixed. Construct a sphere of radius d_1 centered at particle A, as shown in Figure 1. It is clear that particle D has to reside on the surface of this sphere.

Now construct a second sphere of radius d_2 centered at particle B. The rigidity constraint will imply that particle D has to lie on the circle (call it S) formed by the intersection of these two spheres. Note that if the spheres do not intersect, the constraints will not be consistent with the assumption that the distances between particle D and particles A and B are d_1 and d_2 respectively. A third sphere of radius d_3 centered at C will intersect the circle S at two points (D and D' in Figure 1), implying that once the three points are fixed, a fourth particle can be placed at one of only two possible points, consistent with the rigidity constraints given by the fixed distance between particle D and the other three.

The two possible points are related to each other by a reflection about the plane containing particles A, B, and C, and they correspond to the mirror images of each other. But rigid-body displacement excludes reflections, and hence only one of the two points will correspond to the possible position of particle D. Thus there is a unique position where particle D can be placed, and hence no further coordinates need to be specified.

But particle D is completely arbitrary and could be any of the particles in the rigid body other than the original three particles whose positions were specified.

Thus we see that the number of degrees of freedom of a rigid body in three dimensions is the same as that of a rigid body containing three noncollinear particles. Three independent particles have nine degrees of freedom. Since the body is rigid, there are three constraint equations specifying the mutual separations between the three particles. The difference between these two numbers gives the number of degrees of freedom of a rigid body, namely $9 - 3 = 6$.

The next question we address is how to describe the displacement of a rigid body from one configuration to another. There are multiple ways to do this. We shall adopt a scheme that will be convenient in formulating our proof of Chasles's theorem. Consider two configurations I and II of the rigid body. Consider three noncollinear points located at P_1, P_2, P_3 in the rigid body in configuration I. In the final configuration, let these points move to locations P_1', P_2', P_3' respectively. To go from configuration I to configuration II, we will carry out the following three steps:

1. Translate the body by the vector $\overrightarrow{P_1P_1'}$. This ensures that the point at P_1 moves to its final position at P_1'.

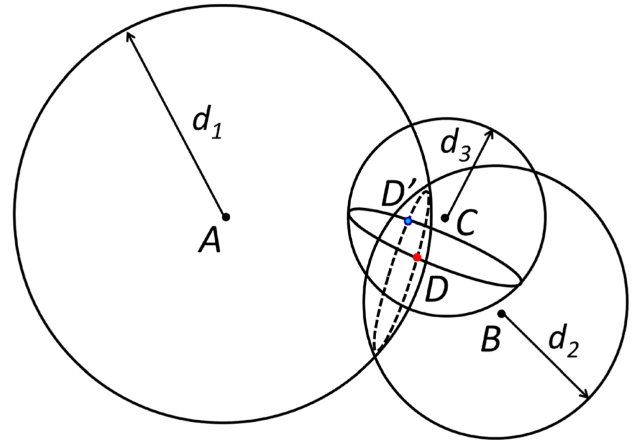

Figure 1. The construction to determine the number of degrees of freedom of a rigid body in three dimensions. If three particles A, B, and C are fixed, then there are only two possible locations for a fourth particle whose distances from the other three are fixed by rigid-body constraints. The possible locations for the fourth particle D are shown as D (red dot) and D' (blue dot) in the figure. They are related by a reflection about the plane containing particles A, B, and C. The dashed circle is the intersection of the spheres centered at A and B (referred to as circle S in the text). Points D and D' are the intersections of this circle with the sphere centered at C.

2. Let P_2'' be the location of the point at P_2 after this translation. Consider the plane formed by the vectors $\overrightarrow{P_1'P_2''}$ and $\overrightarrow{P_1'P_2'}$. This is the equatorial plane shown in Figure 2. Rotate the rigid body about an axis perpendicular to this plane and passing through the point at P_1' such that the point at P_2'' is in its final position at P_2'. Note that if P_2'' is the same as P_2', then this step need not be carried out.
3. Let P_3'' be the location of the original point P_3 after the above two operations. The rigid body can now be rotated about the axis passing through P_1' and P_2' (see Figure 2) such that the point at P_3'' is in its final position at P_3'.

These steps will ensure that the rigid body has been displaced to its final configuration.

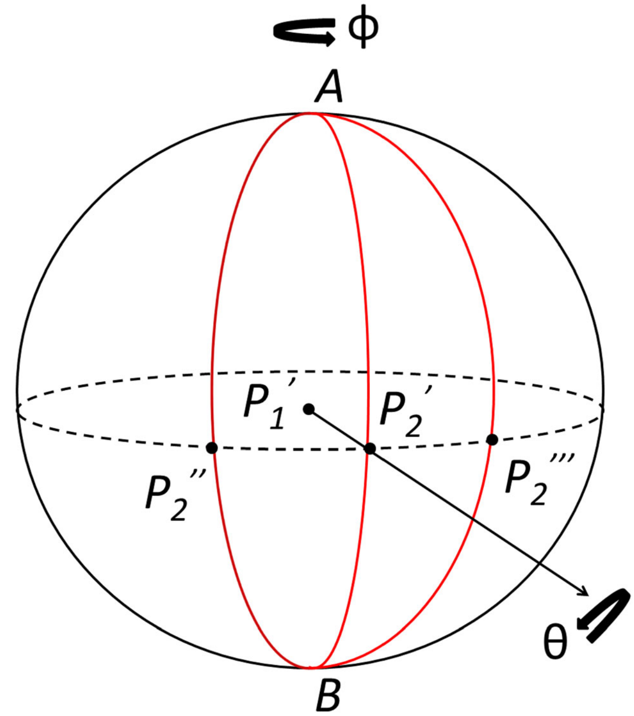

Figure 2. Steps 2 and 3 of the procedure for going from one configuration to the other of the rigid body. The point P_1' is in its final position after step 1. Rotation about the AB-axis (which is perpendicular to the vectors connecting P_1' to P_2'' and P_1' to P_2') by ϕ carries P_2'' to its final position. The second rotation, through angle θ about $\overrightarrow{P_1'P_2'}$, will bring the rigid body to its final configuration. Note that we have oriented the figure such that the AB-axis is vertical. We have not shown P_3'' or P_3' in the figure. These points will not in general lie on the surface of the sphere shown.

Proof of Euler's Rotation Theorem

We are now in a position to prove Euler's rotation theorem. In order to do so, let us look at steps 2 and 3 in the scheme described above to go from one configuration of a rigid body to another. Note that after step 1, the point at P_1 is in its final position, and the point at P_2 has moved over to P_2''. Steps 2 and 3 involve rotations to be carried out with P_1' fixed. These operations are shown in Figure 2. We represent these rotations by R_{AB} and $R_{P_1'P_2'}$. Here R_{AB} is rotation through an angle ϕ about the axis AB (the axis perpendicular to the vectors $\overrightarrow{P_1'P_2''}$ and $\overrightarrow{P_1'P_2'}$ and passing through the point P_1'), which is the axis involved in step 2 above. The value of ϕ can vary between 0 and 2π. Then $R_{P_1'P_2'}$ is rotation through an angle θ about the axis connecting points P_1' and P_2', which corresponds to step 3 above. Here θ can take values from $-\pi$ to π. The sphere shown in the figure has P_1' at its center, and its radius is equal to the distance between points P_1' and P_2''. For convenience, we have oriented the figure such that the AB-axis is vertical.

As the rotation R_{AB} is carried out, the great circle arc $AP_2''B$ will move over into the great circle arc $AP_2'B$. And the entire region that lies between these two arcs before rotation will now lie between the great circle arcs $AP_2'B$ and $AP_2'''B$ (see Figure 2). In particular, the great circle arc ADB that bisects the region $AP_2''BP_2'A$ (see Figure 3) will move over to the great circle arc $AD'B$. Consider now an arc of latitude such as LMN, where L lies on $AP_2''B$, N on $AP_2'B$, and M on ADB. Note that M is the midpoint of the arc. Under rotation R_{AB}, LMN will move over to the latitude arc $NM'N'$. Similarly, the latitude arc HQS (Q being the midpoint) will move over to $SQ'S'$ under rotation (Figure 3).

It is interesting to note what happens to points like M' and Q' under the second rotation (step 3 above). They are candidates for points that could fall back to their original positions after the two rotations! This is so because the great circle arc $P_2'M$ ($P_2'Q$) is equal in magnitude to the great circle arc $P_2'M'$ ($P_2'Q'$), and hence under rotation about an axis passing through P_2' and P_1', both points will fall on the same latitude circle with P_2' as the pole.

We will now argue that depending on the value of θ, there is going to be exactly one such point that will return to its original position (that is, the position before step 2). Figure 4 shows a few of the candidate points that can return to their original locations. It is clear from the figure that the angles shown have the following ordering:

$$DP_2'D' = \pi > MP_2'M' > QP_2'Q' > AP_2'A = 0. \qquad (1)$$

Even though this ordering is apparent from the figure, one can establish it more rigorously in the following manner. The spherical triangle $P_2'DQ$ (see Figure 3) contains the spherical triangle $P_2'DM$. This is because the base $P_2'D$ is common to both triangles, and the great circle arcs DQ and DM are part of the same great circle, with DQ longer than DM. This implies that the angle $NP_2'M$ is larger than $SP_2'Q$ (both being defined as angles between the corresponding great circle arcs). But we have $MP_2'M' = 2MP_2'N$ and $QP_2'Q' = 2QP_2'S$. The relationship in equation (1) follows.

Thus for every positive value of θ in the interval from 0 to π, one and only one of the points of type M' that lie in the hemisphere containing point A will return to its original position. There will be a similar point in the diametrically opposite side of the sphere. If θ were negative and lay

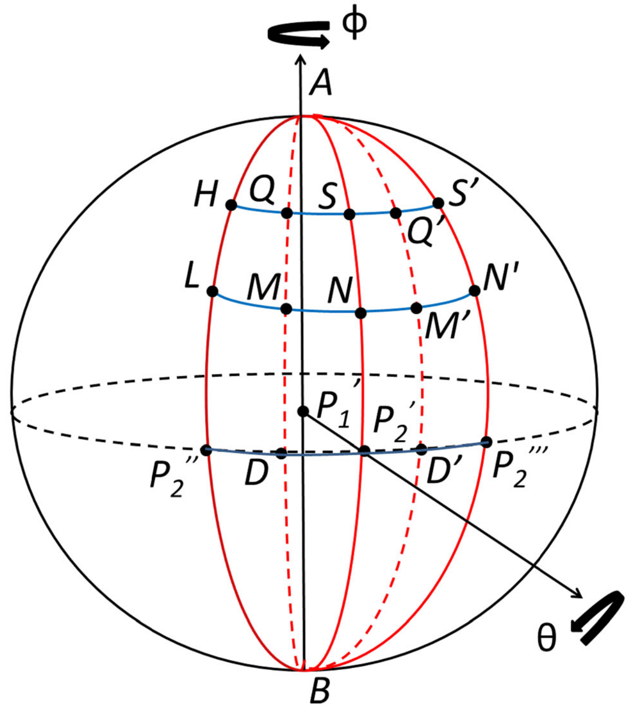

Figure 3. The figure shows how arcs of latitude shift under the rotation about the AB-axis. The equatorial arc $P_2''DP_2'$, D being the midpoint of the arc, moves over to $P_2'D'P_2'''$. The arc LMN, M being the midpoint of the arc, moves over to $NM'N'$. Similarly, point Q, which is the midpoint of the arc of latitude HQS, goes to point Q'. Points like D', M', and Q' can move back to their original positions under the rotation about $P_1'P_2'$ (see Figure 4).

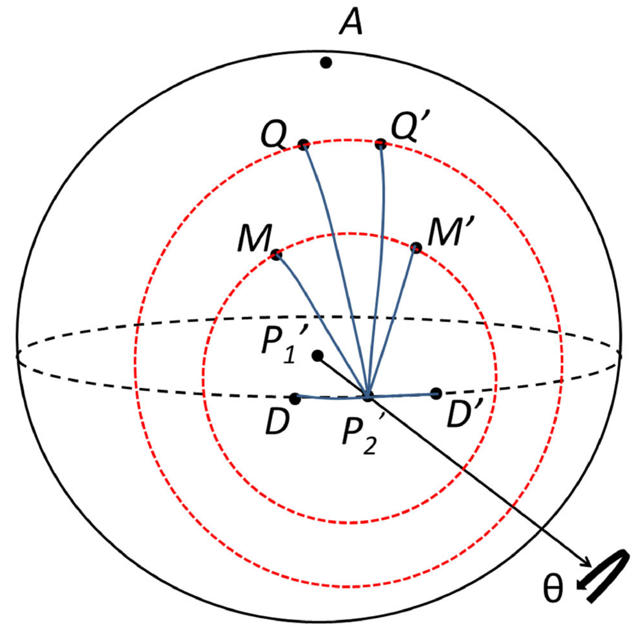

Figure 4. Pairs such as points D' and D, M and M', and Q and Q' are equidistant from point P_2'. This implies that these pairs of points will lie in the same latitudinal circles (red dashed curves) with P_2' as the pole. This, in turn, makes it possible for these points to move back to their original positions (that is, the one prior to rotation about AB) after the rotation about $P_1'P_2'$. In fact, one can show that (see the text) exactly one of these sets of points will fall back to the original position for θ lying between 0 and π. Note that the blue curves in the figure are great circle arcs connecting the points involved.

between 0 and $-\pi$, there would be a point in the lower region below the equatorial plane that would return to its original position and a corresponding point in the diametrically opposite side. Thus for given values of ϕ and θ, there are diametrically opposite pairs of points that do not change position under steps 2 and 3.

This implies that the effect of both the rotations considered above should be the same as that due to a single rotation about an axis that passes through these invariant points and the center of the sphere (P_1'). Since the effect of an arbitrary set of rotations with P_1' fixed can be described using steps 2 and 3 above, we see that the net effect of these rotations can be attained by a single rotation about an axis. This proves Euler's rotation theorem. If we know the value of θ, we can find the invariant point by construction. To find this, choose the point (say X) on the great circle arc ADB such that the angle between the great circle arcs $P_2'X$ and $P_2'A$ is $\theta/2$.

Proof of Chasles's Theorem

We have already shown that the last two steps in our procedure to represent a general displacement of a rigid body correspond to a rotation about a single axis. But step 1 involved a pure translation that took point P_1 to P_1'. Thus we can conclude that a general displacement of the rigid body can be obtained by a translation and a rotation about an axis. To complete the proof of Chasles's theorem, we also need to show that a different choice of point (instead of P_1) for translation will not alter the direction and amount of rotation to be carried out. To prove this, imagine we had chosen a different point Q_1 instead of P_1 for translation. Choose points Q_2 and Q_3 such that $\overrightarrow{P_1P_2} = \overrightarrow{Q_1Q_2}$ and $\overrightarrow{P_1P_3} = \overrightarrow{Q_1Q_3}$, as shown in Figure 5. One can now repeat the arguments above for proving Euler's rotation theorem. Since the vectors involved in steps 2 and 3 ($\overrightarrow{Q_1Q_2}$ and $\overrightarrow{Q_1Q_3}$) in this case are identical to the old ones (even though the new displacement vector could be different), we will end up with the same rotation axis and angle. This completes the proof of Chasles's theorem. It may well be that there is no material point in the rigid body at the location of Q_2 or Q_3. One can nevertheless think of imaginary points rigidly attached to the body and moving in accordance with the rigidity constraints. In fact, the displacing points (such as P_1 and Q_1) themselves need not form part of the rigid body.

An important corollary of Chasles's theorem is the Mozzi–Chasles theorem, which states that the general displacement of a rigid body can be obtained by a rotation about an axis and a translation along the same axis. For completeness, we give here a proof of this theorem.

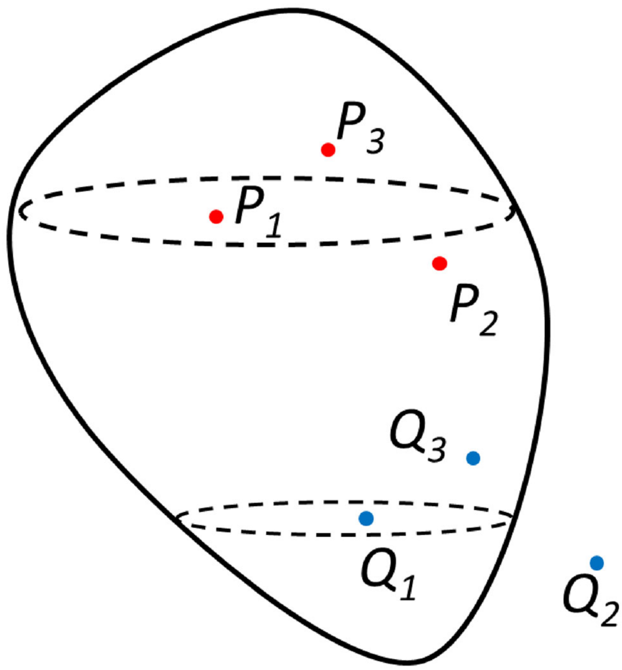

Figure 5. The proof of Chasles's theorem also involves proving that irrespective of the translating point, the direction of the rotation axis and the angle of rotation are the same. If one chooses the translating point to be Q_1 instead of P_1, the constancy in the direction of the axis of rotation and the angle of rotation can be seen by considering points Q_2 and Q_3, which are related to Q_1 as P_2 and P_3 are to P_1.

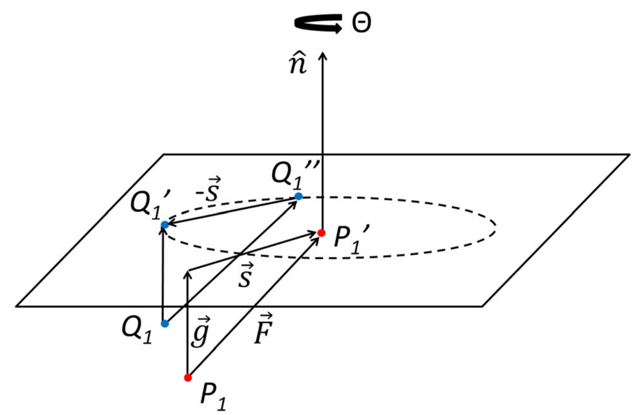

Figure 6. To prove the Mozzi–Chasles theorem, the translation vector connecting P_1 to P_1', that is, $\vec{F}$, is resolved into a part that lies along the axis of rotation ($\vec{g} = g\hat{n}$) and another that lies in the plane perpendicular to it ($\vec{s}$). Under an arbitrary rotation Θ about $\hat{n}$, one can find points that will have a displacement equal to $-\vec{s}$ (for example, displacement of Q_1'' to Q_1' in the figure). Choosing one of these points as the translating point, one can arrive at a translation that is along $\hat{n}$. In the figure, choosing Q_1 as the translating point will ensure that the translation is along the rotation axis itself.

Consider a rigid body displacement described by the displacement vector $\vec{F}$ and the rotation in the direction $\hat{n}$ by an amount Θ, as shown in Figure 6. It may be that $\vec{F}$ is not parallel to $\hat{n}$. We will assume that the rigid displacement affects all the points in space and not just those belonging to the rigid body.

The translation vector $\vec{F}$ can be expressed as the sum of a vector pointing along $\hat{n}$ ($\vec{g}$ in the figure) and a vector lying in the plane perpendicular to $\hat{n}$ ($\vec{s}$ in the figure). Under the rotation about $\hat{n}$, the different points in space will undergo displacements that lie in a plane perpendicular to $\hat{n}$. For every value of Θ, the set of displacement vectors will contain all possible vectors in the plane. This is because the magnitude of the rotation vector will vary from zero to infinity as the distance of the points from the axis of rotation increases from zero to infinity, and all points lying on a circle at fixed distance from the axis of rotation will generate displacements in all possible directions in the plane.

Thus one should be able to find points whose displacement is $-\vec{s}$ under the rotation. If one chooses one of these points as the translating point, it will ensure that the displacement vector is along $\hat{n}$ itself, proving the corollary. The common axis that is involved in this description, in the direction of $\hat{n}$, is referred to as the screw axis or Mozzi axis.

The Mozzi–Chasles theorem leads to another important result concerning the motion of a rigid body in two dimensions. The counterpart of Chasles's theorem in two dimensions, sometimes referred to as Euler's first rotation theorem, states that every displacement of a rigid body in two dimensions can be achieved by either a single rotation or a translation. There is a straightforward geometric proof by construction for this theorem [2]. We shall prove the result using the Mozzi–Chasles theorem. For this, note that in two dimensions, the axis of rotation is always perpendicular to the plane. By the Mozzi–Chasles theorem (since two-dimensional displacements are a subset of possible displacements in three dimensions), the rigid displacement can be achieved by a translation along an axis and rotation about that axis. Since the only possible translation along the rotation axis is one with zero magnitude, there must be a point that does not change its position under rigid displacement in two dimensions. Thus the rigid displacement can then be achieved through a pure rotation about an axis through this point. The other possibility is a pure translation in the plane, in which case the screw axis will lie in the plane and the rotation about the screw axis will be zero. It follows that in two dimensions, a rigid displacement is either a pure translation (screw axis lies in the plane) or a pure rotation (screw axis is normal to the plane).

Concluding Remarks

We have derived Euler's rotation theorem using a novel geometric proof. The proof involves using a set of three steps that takes the rigid body from its initial state to its final state. Euler's rotation theorem was derived using the last of the two steps in this procedure. The proof is presented in a manner that helps in the visualization of how the invariant points arise and is therefore of pedagogical interest. But it should be kept in mind that the sequence

by which one chooses to move from one configuration to another is neither unique nor special.

The first part of Chasles's theorem, which asserts that the general displacement of a rigid body is a combination of translation and a rotation about an axis, follows immediately from Euler's theorem and the first step in the procedure for carrying out rigid displacement. The fact that the orientation of the axis of rotation and angle of rotation are independent of the translation vector involved was proved by a separate construction. For completeness, we have also presented a proof of the existence of a screw axis for motion in three dimensions and a proof that in two dimensions, every rigid displacement can be achieved by a pure rotation or a translation. The proof of Euler's rotation theorem presented here should be accessible to a nonspecialist with some familiarity with high-school geometry. Also, it offers yet another point of view in looking at these important theorems associated with the mechanics of rigid bodies.

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